
Tutorial Sheet-4: Counting, Generating Functions and Recurrence Relations

- (1) How many ways are there to assign three jobs to five employees if each employee can be given more than one job?
- (2) How many ways are there to select three unordered elements from a set with five elements when repetition is allowed?
- (3) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17,$$

where x_1, x_2, x_3 , and x_4 are non-negative integers?

- (4) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i , $i = 1, 2, 3, 4, 5$, is a non-negative integer such that

- (a) $x_1 \geq 1$?
 - (b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?
 - (c) $0 \leq x_1 \leq 10$?
 - (d) $0 \leq x_1 \leq 3$, $1 \leq x_2 < 4$, and $x_3 \geq 15$?
- (5) How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where x_1, x_2 , and x_3 are non-negative integers? [Hint: Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$.]

- (6) Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.
- (8) How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?
- (9) In how many different orders can five runners finish a race if no ties are allowed?
- (10) In how many ways can a set of five letters be selected from the English alphabet?
- (11) How many subsets with more than two elements does a set with 100 elements have?
- (12) (a) What is the generating function for $\{a_k\}$, where a_k is the number of solutions of

$$x_1 + x_2 + x_3 + x_4 = k,$$

when x_1, x_2, x_3 , and x_4 are integers with $x_1 \geq 3$, $1 \leq x_2 \leq 5$, $0 \leq x_3 \leq 4$, and $x_4 \geq 1$?

- (b) Use your answer to part (a) to find a_7 ?

- (13) Use generating functions to solve the recurrence relation $a_k = 7a_{k-1}$ with the initial condition $a_0 = 5$.
- (14) Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.
- (15) Solve these recurrence relations together with the initial conditions given
- (a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$
 - (b) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$
 - (c) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = -3$
 - (d) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0$, $a_0 = 2$, $a_1 = 8$.